Addendum to AC Power Flows and their Derivatives using Complex Matrix Notation: Nodal Current Balance

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MATPOWER *Technical Note 3*

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Contents

1 Notation

n_b, n_g, n_l	number of buses, generators, branches, respectively
$\|v_i\|, \theta_i$	bus voltage magnitude and angle at bus i
v_i	complex bus voltage at bus i, that is $\|v_i\|e^{j\theta_i}$
$\mathcal{V}, \Theta$	$n_b \times 1$ vectors of bus voltage magnitudes and angles
V, Λ	$n_b \times 1$ vector of complex bus voltages v_i and their inverses $\frac{1}{v_i}$
I^{bus}	$n_b \times 1$ vector of complex bus current injections
I^f, I^t	$n_l \times 1$ vectors of complex branch current injections, *from* and *to* ends
S^{bus}	$n_b \times 1$ vector of complex bus power injections
S^f, S^t	$n_l \times 1$ vectors of complex branch power flows, *from* and *to* ends
S_g	$n_g \times 1$ vector of generator complex power injections
P, Q	real and reactive power flows/injections, $S = P + jQ$
M, N	real and imaginary parts of current flows/injections, $I = M + jN$
Y_{bus}	$n_b \times n_b$ system bus admittance matrix
Y_f, Y_t	$n_l \times n_b$ system branch admittance matrices, *from* and *to* ends
C_g	$n_b \times n_g$ generator connection matrix $(i,j)^{th}$ element is 1 if generator j is located at bus i, 0 otherwise
C_f, C_t	$n_l \times n_b$ branch connection matrices, *from* and *to* ends, $(i,j)^{th}$ element is 1 if *from* end, or *to* end, respectively, of branch i is connected to bus j, 0 otherwise
$[A]$	diagonal matrix with vector A on the diagonal
A^T	(non-conjugate) transpose of matrix A
A^*	complex conjugate of A
A^b	matrix exponent for matrix A, or element-wise exponent for vector A
$\mathbf{1}_n, [\mathbf{1}_n]$	$n \times 1$ vector of all ones, $n \times n$ identity matrix
$\mathbf{0}$	appropriately-sized vector or matrix of all zeros

2 Introduction

This document is a supplement to MATPOWER *Technical Note 2* "AC Power Flows, Generalized OPF Costs and their Derivatives using Complex Matrix Notation" [1], adding formulas for full nodal current balance, including injections from generators and loads. MATPOWER *Technical Note 4* [2] presents formulas for variations based on a cartesian coordinate representation of bus voltages.

The purpose of these documents is to show how the AC power balance and flow equations used in power flow and optimal power flow computations can be expressed in terms of complex matrices, and how their first and second derivatives can be computed efficiently using complex sparse matrix manipulations. The relevant code in MATPOWER [3–5] is based on the formulas found in these three notes.

We will be looking at complex functions of the real valued vector

$$X = \begin{bmatrix} \Theta \\ \mathcal{V} \\ P_g \\ Q_g \end{bmatrix}. \tag{1}$$

For a complex scalar function $f\colon \mathbb{R}^n \to \mathbb{C}$ of a real vector $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\mathsf{T}$, we use the following notation for the first derivatives (transpose of the gradient)

$$f_X = \frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}. \tag{2}$$

The matrix of second partial derivatives, the Hessian of f, is

$$f_{XX} = \frac{\partial^2 f}{\partial X^2} = \frac{\partial}{\partial X}\left(\frac{\partial f}{\partial X}\right)^\mathsf{T} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}. \tag{3}$$

For a complex vector function $F\colon \mathbb{R}^n \to \mathbb{C}^m$ of a vector X, where

$$F(X) = \begin{bmatrix} f_1(X) & f_2(X) & \cdots & f_m(X) \end{bmatrix}^\mathsf{T}, \tag{4}$$

the first derivatives form the Jacobian matrix, where row i is the transpose of the gradient of f_i.

$$F_X = \frac{\partial F}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{5}$$

In these derivations, the full 3-dimensional set of second partial derivatives of F will not be computed. Instead a matrix of partial derivatives will be formed by computing the Jacobian of the vector function obtained by multiplying the transpose of the Jacobian of F by a constant vector λ, using the following notation.

$$F_{XX}(\alpha) = \left(\frac{\partial}{\partial X} \left({F_X}^{\mathsf{T}} \lambda \right) \right) \bigg|_{\lambda=\alpha} \tag{6}$$

Just to clarify the notation, if Y and Z are subvectors of X, then

$$F_{YZ}(\alpha) = \left(\frac{\partial}{\partial Z} \left({F_Y}^{\mathsf{T}} \lambda \right) \right) \bigg|_{\lambda=\alpha}. \tag{7}$$

One common operation encountered in these derivations is the element-wise multiplication of a vector A by a vector B to form a new vector C of the same dimension, which can be expressed in either of the following forms

$$C = [A]\, B = [B]\, A \tag{8}$$

It is useful to note that the derivative of such a vector can be calculated by the chain rule as

$$C_X = \frac{\partial C}{\partial X} = [A] \frac{\partial B}{\partial X} + [B] \frac{\partial A}{\partial X} = [A]\, B_X + [B]\, A_X \tag{9}$$

3 Voltages

3.1 Bus Voltages

See the corresponding section in MATPOWER *Technical Note 2*. Consider also the vector of inverses of bus voltages $\frac{1}{v_i}$, denoted by Λ. Note that

$$\frac{1}{v_i} = \frac{1}{|v_i| e^{j\theta_i}} = \frac{|v_i| e^{-j\theta_i}}{|v_i|^2} = \frac{{v_i}^*}{|v_i|^2} \tag{10}$$

$$\Lambda = V^{-1} = [\mathcal{V}]^{-2}\, V^* \tag{11}$$

3.1.1 First Derivatives

$$\Lambda_\Theta = \frac{\partial \Lambda}{\partial \Theta} = -\left[V\right]^{-2} V_\Theta = -j\left[V\right]^{-1} = -j\left[\Lambda\right] \tag{12}$$

$$\Lambda_\mathcal{V} = \frac{\partial \Lambda}{\partial \mathcal{V}} = -\left[V\right]^{-2} V_\mathcal{V} = -\left[V\right]^{-1} \left[\mathcal{V}\right]^{-1} = -\left[\mathcal{V}\right]^{-1} \left[\Lambda\right] \tag{13}$$

4 Bus Injections

4.1 Complex Current Injections

Consider the complex current balance equation, $G^c(X) = \mathbf{0}$, where

$$G^c(X) = I^{\text{bus}} + I^{dg} \tag{14}$$

and

$$I^{\text{bus}} = Y_{\text{bus}} V \tag{15}$$
$$I^{dg} = [S_d - C_g S_g]^* \Lambda^* \tag{16}$$

4.1.1 First Derivatives

$$I_X^{\text{bus}} = \frac{\partial I^{\text{bus}}}{\partial X} = \begin{bmatrix} I_\Theta^{\text{bus}} & I_\mathcal{V}^{\text{bus}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{17}$$

$$I_\Theta^{\text{bus}} = \frac{\partial I^{\text{bus}}}{\partial \Theta} = Y_{\text{bus}} \frac{\partial V}{\partial \Theta} = j Y_{\text{bus}} [V] \tag{18}$$

$$I_\mathcal{V}^{\text{bus}} = \frac{\partial I^{\text{bus}}}{\partial \mathcal{V}} = Y_{\text{bus}} \frac{\partial V}{\partial \mathcal{V}} = Y_{\text{bus}} [V] [\mathcal{V}]^{-1} = Y_{\text{bus}} [E] \tag{19}$$

$$I_X^{dg} = \frac{\partial I^{dg}}{\partial X} = \begin{bmatrix} I_\Theta^{dg} & I_\mathcal{V}^{dg} & I_{P_g}^{dg} & I_{Q_g}^{dg} \end{bmatrix} \tag{20}$$

$$I_\Theta^{dg} = \frac{\partial I^{dg}}{\partial \Theta} = j[S_d - C_g S_g]^* [\Lambda^*] \tag{21}$$

$$I_\mathcal{V}^{dg} = \frac{\partial I^{dg}}{\partial \mathcal{V}} = -[S_d - C_g S_g]^* [\mathcal{V}]^{-1} [\Lambda^*] \tag{22}$$

$$I_{P_g}^{dg} = \frac{\partial I^{dg}}{\partial P_g} = -[\Lambda^*] C_g \tag{23}$$

$$I_{Q_g}^{dg} = \frac{\partial I^{dg}}{\partial Q_g} = j [\Lambda^*] C_g \tag{24}$$

$$G_X^c = \frac{\partial G^c}{\partial X} = \begin{bmatrix} G_\Theta^c & G_\mathcal{V}^c & G_{P_g}^c & G_{Q_g}^c \end{bmatrix} \tag{25}$$

$$G_\Theta^c = \frac{\partial G^c}{\partial \Theta} = I_\Theta^{\text{bus}} + I_\Theta^{dg} = j\left(Y_{\text{bus}}\left[V\right] + \left[S_d - C_g S_g\right]^* \left[\Lambda^*\right]\right) \tag{26}$$

$$G_\mathcal{V}^c = \frac{\partial G^c}{\partial \mathcal{V}} = I_\mathcal{V}^{\text{bus}} + I_\mathcal{V}^{dg} = Y_{\text{bus}}\left[E\right] - \left[S_d - C_g S_g\right]^* \left[\mathcal{V}\right]^{-1} \left[\Lambda^*\right] \tag{27}$$

$$G_{P_g}^c = \frac{\partial G^c}{\partial P_g} = I_{P_g}^{dg} = -\left[\Lambda^*\right] C_g \tag{28}$$

$$G_{Q_g}^c = \frac{\partial G^c}{\partial Q_g} = I_{Q_g}^{dg} = j\left[\Lambda^*\right] C_g \tag{29}$$

4.1.2 Second Derivatives

$$I_{XX}^{\text{bus}}(\lambda) = \frac{\partial}{\partial X}\left({I_X^{\text{bus}}}^\mathsf{T} \lambda\right) \tag{30}$$

$$= \begin{bmatrix} I_{\Theta\Theta}^{\text{bus}}(\lambda) & I_{\Theta\mathcal{V}}^{\text{bus}}(\lambda) & \mathbf{0} & \mathbf{0} \\ I_{\mathcal{V}\Theta}^{\text{bus}}(\lambda) & I_{\mathcal{V}\mathcal{V}}^{\text{bus}}(\lambda) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} \mathcal{B} & \mathcal{C} & \mathbf{0} & \mathbf{0} \\ \mathcal{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{32}$$

$$I_{\Theta\Theta}^{\text{bus}}(\lambda) = \frac{\partial}{\partial \Theta}\left({I_\Theta^{\text{bus}}}^\mathsf{T} \lambda\right) \tag{33}$$

$$= \frac{\partial}{\partial \Theta}\left(j\left[V\right] {Y_{\text{bus}}}^\mathsf{T} \lambda\right) \tag{34}$$

$$= j\left[{Y_{\text{bus}}}^\mathsf{T} \lambda\right] V_\Theta \tag{35}$$

$$= -\left[{Y_{\text{bus}}}^\mathsf{T} \lambda\right]\left[V\right] \tag{36}$$

$$= \mathcal{B} \tag{37}$$

$$
\begin{aligned}
I^{\rm bus}_{\mathcal{V}\Theta}(\lambda) &= \frac{\partial}{\partial \Theta}\left({I^{\rm bus}_{\mathcal{V}}}^{\mathsf{T}} \lambda \right) & (38)\\
&= \frac{\partial}{\partial \Theta}\left([E]\, {Y_{\rm bus}}^{\mathsf{T}} \lambda \right) & (39)\\
&= \left[{Y_{\rm bus}}^{\mathsf{T}} \lambda \right] E_\Theta & (40)\\
&= j \left[{Y_{\rm bus}}^{\mathsf{T}} \lambda \right] [E] & (41)\\
&= \mathcal{C} & (42)
\end{aligned}
$$

$$
\begin{aligned}
I^{\rm bus}_{\Theta\mathcal{V}}(\lambda) &= \frac{\partial}{\partial \mathcal{V}}\left({I^{\rm bus}_{\Theta}}^{\mathsf{T}} \lambda \right) & (43)\\
&= \frac{\partial}{\partial \mathcal{V}}\left(j\, [V]\, {Y_{\rm bus}}^{\mathsf{T}} \lambda \right) & (44)\\
&= j \left[{Y_{\rm bus}}^{\mathsf{T}} \lambda \right] V_{\mathcal{V}} & (45)\\
&= j \left[{Y_{\rm bus}}^{\mathsf{T}} \lambda \right] [E] & (46)\\
&= {I^{\rm bus}_{\mathcal{V}\Theta}}^{\mathsf{T}}(\lambda) = \mathcal{C} & (47)
\end{aligned}
$$

$$
\begin{aligned}
I^{\rm bus}_{\mathcal{V}\mathcal{V}}(\lambda) &= \frac{\partial}{\partial \mathcal{V}}\left({I^{\rm bus}_{\mathcal{V}}}^{\mathsf{T}} \lambda \right) & (48)\\
&= \frac{\partial}{\partial \mathcal{V}}\left([E]\, {Y_{\rm bus}}^{\mathsf{T}} \lambda \right) & (49)\\
&= \left[{Y_{\rm bus}}^{\mathsf{T}} \lambda \right] E_{\mathcal{V}} & (50)\\
&= \mathbf{0} & (51)
\end{aligned}
$$

$$
\begin{aligned}
I^{dg}_{XX}(\lambda) &= \frac{\partial}{\partial X}\left({I^{dg}_{X}}^{\mathsf{T}} \lambda \right) & (52)\\
&= \left[\begin{array}{cccc}
I^{dg}_{\Theta\Theta}(\lambda) & I^{dg}_{\Theta\mathcal{V}}(\lambda) & I^{dg}_{\Theta P_g}(\lambda) & I^{dg}_{\Theta Q_g}(\lambda) \\
I^{dg}_{\mathcal{V}\Theta}(\lambda) & I^{dg}_{\mathcal{V}\mathcal{V}}(\lambda) & I^{dg}_{\mathcal{V} P_g}(\lambda) & I^{dg}_{\mathcal{V} Q_g}(\lambda) \\
I^{dg}_{P_g\Theta}(\lambda) & I^{dg}_{P_g\mathcal{V}}(\lambda) & \mathbf{0} & \mathbf{0} \\
I^{dg}_{Q_g\Theta}(\lambda) & I^{dg}_{Q_g\mathcal{V}}(\lambda) & \mathbf{0} & \mathbf{0}
\end{array} \right] & (53)\\
&= \left[\begin{array}{cccc}
-\mathcal{G} & -j\mathcal{H} & -j\mathcal{K}^{\mathsf{T}} & -\mathcal{K}^{\mathsf{T}} \\
-j\mathcal{H} & 2\mathcal{D}\mathcal{H} & \mathcal{L}^{\mathsf{T}} & -j\mathcal{L}^{\mathsf{T}} \\
-j\mathcal{K} & \mathcal{L} & \mathbf{0} & \mathbf{0} \\
-\mathcal{K} & -j\mathcal{L} & \mathbf{0} & \mathbf{0}
\end{array} \right] & (54)
\end{aligned}
$$

$$I^{dg}_{\Theta\Theta}(\lambda) = \frac{\partial}{\partial\Theta}\left({I^{dg}_{\Theta}}^{\mathsf{T}}\lambda\right) \tag{55}$$
$$= \frac{\partial}{\partial\Theta}\left(j[S_d - C_gS_g]^*\,[\Lambda^*]\,\lambda\right) \tag{56}$$
$$= j[S_d - C_gS_g]^*\,[\lambda]\,\Lambda^*_{\Theta} \tag{57}$$
$$= -[S_d - C_gS_g]^*\,[\lambda]\,[\Lambda^*] \tag{58}$$
$$= -\mathcal{G} \tag{59}$$

$$I^{dg}_{\mathcal{V}\Theta}(\lambda) = \frac{\partial}{\partial\Theta}\left({I^{dg}_{\mathcal{V}}}^{\mathsf{T}}\lambda\right) \tag{60}$$
$$= \frac{\partial}{\partial\Theta}\left(-[S_d - C_gS_g]^*\,[\mathcal{V}]^{-1}\,[\Lambda^*]\,\lambda\right) \tag{61}$$
$$= -[S_d - C_gS_g]^*\,[\lambda]\,[\mathcal{V}]^{-1}\,\Lambda^*_{\Theta} \tag{62}$$
$$= -j[S_d - C_gS_g]^*\,[\lambda]\,[\mathcal{V}]^{-1}\,[\Lambda^*] \tag{63}$$
$$= -j\mathcal{H} \tag{64}$$

$$I^{dg}_{P_g\Theta}(\lambda) = \frac{\partial}{\partial\Theta}\left({I^{dg}_{P_g}}^{\mathsf{T}}\lambda\right) \tag{65}$$
$$= \frac{\partial}{\partial\Theta}\left(-{C_g}^{\mathsf{T}}\,[\Lambda^*]\,\lambda\right) \tag{66}$$
$$= -{C_g}^{\mathsf{T}}\,[\lambda]\,\Lambda^*_{\Theta} \tag{67}$$
$$= -j{C_g}^{\mathsf{T}}\,[\lambda]\,[\Lambda^*] \tag{68}$$
$$= -j\mathcal{K} \tag{69}$$

$$I^{dg}_{Q_g\Theta}(\lambda) = \frac{\partial}{\partial\Theta}\left({I^{dg}_{Q_g}}^{\mathsf{T}}\lambda\right) \tag{70}$$
$$= \frac{\partial}{\partial\Theta}\left(j{C_g}^{\mathsf{T}}\,[\Lambda^*]\,\lambda\right) \tag{71}$$
$$= j{C_g}^{\mathsf{T}}\,[\lambda]\,\Lambda^*_{\Theta} \tag{72}$$
$$= -{C_g}^{\mathsf{T}}\,[\lambda]\,[\Lambda^*] \tag{73}$$
$$= -\mathcal{K} \tag{74}$$

$$I^{dg}_{\Theta\mathcal{V}}(\lambda) = \frac{\partial}{\partial\mathcal{V}}\left({I^{dg}_{\Theta}}^{\mathsf{T}}\lambda\right) \tag{75}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial \mathcal{V}} \left(j[S_d - C_g S_g]^* [\Lambda^*] \lambda \right) && (76)\\
&= j[S_d - C_g S_g]^* [\lambda] \Lambda^*_{\mathcal{V}} && (77)\\
&= -j[S_d - C_g S_g]^* [\lambda] [\mathcal{V}]^{-1} [\Lambda^*] && (78)\\
&= {I^{dg}_{\mathcal{V}\Theta}}^{\mathsf{T}}(\lambda) = -j\mathcal{H} && (79)
\end{aligned}
$$

$$
\begin{aligned}
I^{dg}_{\mathcal{V}\mathcal{V}}(\lambda) &= \frac{\partial}{\partial \mathcal{V}} \left({I^{dg}_{\mathcal{V}}}^{\mathsf{T}} \lambda \right) && (80)\\
&= \frac{\partial}{\partial \mathcal{V}} \left(-[S_d - C_g S_g]^* [\mathcal{V}]^{-1} [\Lambda^*] \lambda \right) && (81)\\
&= -[S_d - C_g S_g]^* [\lambda] \left([\mathcal{V}]^{-1} \Lambda^*_{\mathcal{V}} + [\Lambda^*] \frac{\partial \mathcal{V}^{-1}}{\partial \mathcal{V}} \right) && (82)\\
&= -[S_d - C_g S_g]^* [\lambda] \left([\mathcal{V}]^{-1} \left(-[\mathcal{V}]^{-1} [\Lambda^*] \right) - [\Lambda^*] [\mathcal{V}]^{-2} \right) && (83)\\
&= 2[S_d - C_g S_g]^* [\lambda] [\mathcal{V}]^{-2} [\Lambda^*] && (84)\\
&= 2\mathcal{D}\mathcal{H} && (85)
\end{aligned}
$$

$$
\begin{aligned}
I^{dg}_{P_g\mathcal{V}}(\lambda) &= \frac{\partial}{\partial \mathcal{V}} \left({I^{dg}_{P_g}}^{\mathsf{T}} \lambda \right) && (86)\\
&= \frac{\partial}{\partial \mathcal{V}} \left(-{C_g}^{\mathsf{T}} [\Lambda^*] \lambda \right) && (87)\\
&= -{C_g}^{\mathsf{T}} [\lambda] \Lambda^*_{\mathcal{V}} && (88)\\
&= {C_g}^{\mathsf{T}} [\lambda] [\mathcal{V}]^{-1} [\Lambda^*] && (89)\\
&= \mathcal{L} && (90)
\end{aligned}
$$

$$
\begin{aligned}
I^{dg}_{Q_g\mathcal{V}}(\lambda) &= \frac{\partial}{\partial \mathcal{V}} \left({I^{dg}_{Q_g}}^{\mathsf{T}} \lambda \right) && (91)\\
&= \frac{\partial}{\partial \mathcal{V}} \left(j{C_g}^{\mathsf{T}} [\Lambda^*] \lambda \right) && (92)\\
&= j{C_g}^{\mathsf{T}} [\lambda] \Lambda^*_{\mathcal{V}} && (93)\\
&= -j{C_g}^{\mathsf{T}} [\lambda] [\mathcal{V}]^{-1} [\Lambda^*] && (94)\\
&= -j\mathcal{L} && (95)
\end{aligned}
$$

$$
\begin{aligned}
I^{dg}_{\Theta P_g}(\lambda) &= \frac{\partial}{\partial P_g}\left({I^{dg}_{\Theta}}^{\mathsf{T}}\lambda\right) && (96)\\
&= \frac{\partial}{\partial P_g}\left(j[S_d - C_g S_g]^*\,[\Lambda^*]\,\lambda\right) && (97)\\
&= -j\,[\lambda]\,[\Lambda^*]\,C_g && (98)\\
&= {I^{dg}_{P_g\Theta}}^{\mathsf{T}}(\lambda) = -j\mathcal{K}^{\mathsf{T}} && (99)
\end{aligned}
$$

$$
\begin{aligned}
I^{dg}_{\mathcal{V}P_g}(\lambda) &= \frac{\partial}{\partial P_g}\left({I^{dg}_{\mathcal{V}}}^{\mathsf{T}}\lambda\right) && (100)\\
&= \frac{\partial}{\partial P_g}\left(-[S_d - C_g S_g]^*\,[\mathcal{V}]^{-1}\,[\Lambda^*]\,\lambda\right) && (101)\\
&= [\lambda]\,[\mathcal{V}]^{-1}\,[\Lambda^*]\,C_g && (102)\\
&= {I^{dg}_{P_g\mathcal{V}}}^{\mathsf{T}}(\lambda) = \mathcal{L}^{\mathsf{T}} && (103)
\end{aligned}
$$

$$
\begin{aligned}
I^{dg}_{\Theta Q_g}(\lambda) &= \frac{\partial}{\partial Q_g}\left({I^{dg}_{\Theta}}^{\mathsf{T}}\lambda\right) && (104)\\
&= \frac{\partial}{\partial Q_g}\left(j[S_d - C_g S_g]^*\,[\Lambda^*]\,\lambda\right) && (105)\\
&= -\,[\lambda]\,[\Lambda^*]\,C_g && (106)\\
&= {I^{dg}_{Q_g\Theta}}^{\mathsf{T}}(\lambda) = -\mathcal{K}^{\mathsf{T}} && (107)
\end{aligned}
$$

$$
\begin{aligned}
I^{dg}_{\mathcal{V}Q_g}(\lambda) &= \frac{\partial}{\partial Q_g}\left({I^{dg}_{\mathcal{V}}}^{\mathsf{T}}\lambda\right) && (108)\\
&= \frac{\partial}{\partial Q_g}\left(-[S_d - C_g S_g]^*\,[\mathcal{V}]^{-1}\,[\Lambda^*]\,\lambda\right) && (109)\\
&= -j\,[\lambda]\,[\mathcal{V}]^{-1}\,[\Lambda^*]\,C_g && (110)\\
&= {I^{dg}_{Q_g\mathcal{V}}}^{\mathsf{T}}(\lambda) = -j\mathcal{L}^{\mathsf{T}} && (111)
\end{aligned}
$$

$$
G^c_{XX}(\lambda) = \frac{\partial}{\partial X}\left({G^c_X}^{\mathsf{T}}\lambda\right) \tag{112}
$$

$$
= \begin{bmatrix} G^c_{\Theta\Theta}(\lambda) & G^c_{\Theta\mathcal{V}}(\lambda) & G^c_{\Theta P_g}(\lambda) & G^c_{\Theta Q_g}(\lambda) \\ G^c_{\mathcal{V}\Theta}(\lambda) & G^c_{\mathcal{V}\mathcal{V}}(\lambda) & G^c_{\mathcal{V}P_g}(\lambda) & G^c_{\mathcal{V}Q_g}(\lambda) \\ G^c_{P_g\Theta}(\lambda) & G^c_{P_g\mathcal{V}}(\lambda) & \mathbf{0} & \mathbf{0} \\ G^c_{Q_g\Theta}(\lambda) & G^c_{Q_g\mathcal{V}}(\lambda) & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{113}
$$

$$
= I^{\text{bus}}_{XX}(\lambda) + I^{dg}_{XX}(\lambda) \tag{114}
$$

$$
= \begin{bmatrix} \mathcal{B} - \mathcal{G} & \mathcal{C} - j\mathcal{H} & -j\mathcal{K}^\mathsf{T} & -\mathcal{K}^\mathsf{T} \\ \mathcal{C} - j\mathcal{H} & 2\mathcal{D}\mathcal{H} & \mathcal{L}^\mathsf{T} & -j\mathcal{L}^\mathsf{T} \\ -j\mathcal{K} & \mathcal{L} & \mathbf{0} & \mathbf{0} \\ -\mathcal{K} & -j\mathcal{L} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{115}
$$

Computational savings can be achieved by storing and reusing certain intermediate terms during the computation of these second derivatives, as follows:

$$
\mathcal{A} = \left[{Y_{\text{bus}}}^\mathsf{T} \lambda\right] \tag{116}
$$

$$
\mathcal{B} = -\mathcal{A}\left[V\right] \tag{117}
$$

$$
\mathcal{C} = j\mathcal{A}\left[E\right] \tag{118}
$$

$$
\mathcal{D} = \left[\mathcal{V}\right]^{-1} \tag{119}
$$

$$
\mathcal{E} = \left[\lambda\right]\left[\Lambda^*\right] \tag{120}
$$

$$
\mathcal{F} = \left[S_d - C_g S_g\right]^* \tag{121}
$$

$$
\mathcal{G} = \mathcal{E}\mathcal{F} \tag{122}
$$

$$
\mathcal{H} = \mathcal{D}\mathcal{G} \tag{123}
$$

$$
\mathcal{K} = {C_g}^\mathsf{T} \mathcal{E} \tag{124}
$$

$$
\mathcal{L} = \mathcal{K}\mathcal{D} \tag{125}
$$

$$
G^c_{\Theta\Theta}(\lambda) = \mathcal{B} - \mathcal{G} \tag{126}
$$

$$
G^c_{\mathcal{V}\Theta}(\lambda) = \mathcal{C} - j\mathcal{H} \tag{127}
$$

$$
G^c_{P_g\Theta}(\lambda) = -j\mathcal{K} \tag{128}
$$

$$
G^c_{Q_g\Theta}(\lambda) = -\mathcal{K} \tag{129}
$$

$$
G^c_{\mathcal{V}\mathcal{V}}(\lambda) = 2\mathcal{D}\mathcal{H} \tag{130}
$$

$$
G^c_{P_g\mathcal{V}}(\lambda) = \mathcal{L} \tag{131}
$$

$$
G^c_{Q_g\mathcal{V}}(\lambda) = -j\mathcal{L} \tag{132}
$$

$$
G^c_{\Theta\mathcal{V}}(\lambda) = G^c_{\mathcal{V}\Theta}(\lambda) \tag{133}
$$

$$
G^c_{\Theta P_g}(\lambda) = {G^c_{P_g\Theta}}^\mathsf{T}(\lambda) \tag{134}
$$

$$G^c_{\mathcal{V}P_g}(\lambda) = {G^c_{P_g\mathcal{V}}}^{\mathsf{T}}(\lambda) \tag{135}$$
$$G^c_{\Theta Q_g}(\lambda) = {G^c_{Q_g\Theta}}^{\mathsf{T}}(\lambda) \tag{136}$$
$$G^c_{\mathcal{V}Q_g}(\lambda) = {G^c_{Q_g\mathcal{V}}}^{\mathsf{T}}(\lambda) \tag{137}$$

4.2 Complex Power Injections

See the corresponding section in MATPOWER *Technical Note 2*.

5 Branch Flows

See the corresponding section in MATPOWER *Technical Note 2*.

6 Generalized AC OPF Costs

Let X be defined as in MATPOWER *Technical Note 2*

$$X = \begin{bmatrix} \Theta \\ \mathcal{V} \\ P_g \\ Q_g \\ Y \\ Z \end{bmatrix} \tag{138}$$

where Y is the $n_y \times 1$ vector of cost variables associated with piecewise linear generator costs and Z is an $n_z \times 1$ vector of additional linearly constrained user variables.

See the corresponding section in MATPOWER *Technical Note 2* for additional details.

7 Lagrangian of the AC OPF

Consider the following AC OPF problem formulation, where X is defined as in (138), f is the generalized cost function described above, and $\mathcal{X}$ represents the reduced form of X, consisting of only Θ, $\mathcal{V}$, P_g and Q_g, without Y and Z.

$$\min_X f(X) \tag{139}$$

subject to

$$G(X) = \mathbf{0} \tag{140}$$
$$H(X) \leq \mathbf{0} \tag{141}$$

where

$$G(X) = \begin{bmatrix} \Re\{G^c(\mathcal{X})\} \\ \Im\{G^c(\mathcal{X})\} \\ A_E X - B_E \end{bmatrix} \tag{142}$$

and

$$H(X) = \begin{bmatrix} H^f(\mathcal{X}) \\ H^t(\mathcal{X}) \\ A_I X - B_I \end{bmatrix} \tag{143}$$

Partitioning the corresponding multipliers λ and μ similarly,

$$\lambda = \begin{bmatrix} \lambda_M \\ \lambda_N \\ \lambda_E \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_f \\ \mu_t \\ \mu_I \end{bmatrix} \tag{144}$$

the Lagrangian for this problem can be written as

$$\mathcal{L}(X, \lambda, \mu) = f(X) + \lambda^\mathsf{T} G(X) + \mu^\mathsf{T} H(X) \tag{145}$$

7.1 Nodal Current Balance

7.1.1 First Derivatives

$$\mathcal{L}_X(X, \lambda, \mu) = f_X + \lambda^\mathsf{T} G_X + \mu^\mathsf{T} H_X \tag{146}$$
$$\mathcal{L}_\lambda(X, \lambda, \mu) = G^\mathsf{T}(X) \tag{147}$$
$$\mathcal{L}_\mu(X, \lambda, \mu) = H^\mathsf{T}(X) \tag{148}$$

where

$$G_X = \begin{bmatrix} \Re\{G^c_\mathcal{X}\} & \mathbf{0} & \mathbf{0} \\ \Im\{G^c_\mathcal{X}\} & \mathbf{0} & \mathbf{0} \\ & A_E & \end{bmatrix} = \begin{bmatrix} \Re\{G^c_\Theta\} & \Re\{G^c_\mathcal{V}\} & \Re\{G^c_{P_g}\} & \Re\{G^c_{Q_g}\} & \mathbf{0} & \mathbf{0} \\ \Im\{G^c_\Theta\} & \Im\{G^c_\mathcal{V}\} & \Im\{G^c_{P_g}\} & \Im\{G^c_{Q_g}\} & \mathbf{0} & \mathbf{0} \\ & & A_E & & & \end{bmatrix} \tag{149}$$

and

$$H_X = \begin{bmatrix} H^f_\mathcal{X} & \mathbf{0} & \mathbf{0} \\ H^t_\mathcal{X} & \mathbf{0} & \mathbf{0} \\ & A_I & \end{bmatrix} = \begin{bmatrix} H^f_\Theta & H^f_\mathcal{V} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ H^t_\Theta & H^t_\mathcal{V} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & A_I & & & \end{bmatrix} \tag{150}$$

7.1.2 Second Derivatives

$$\mathcal{L}_{XX}(X,\lambda,\mu) = f_{XX} + G_{XX}(\lambda) + H_{XX}(\mu) \tag{151}$$

where

$$G_{XX}(\lambda) = \begin{bmatrix} \Re\{G^c_{\mathcal{X}\mathcal{X}}(\lambda_M)\} + \Im\{G^c_{\mathcal{X}\mathcal{X}}(\lambda_N)\} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{152}$$

$$\begin{aligned} = \Re\left\{\begin{bmatrix} G^c_{\Theta\Theta}(\lambda_M) & G^c_{\Theta\mathcal{V}}(\lambda_M) & G^c_{\Theta P_g}(\lambda_M) & G^c_{\Theta Q_g}(\lambda_M) & \mathbf{0} & \mathbf{0} \\ G^c_{\mathcal{V}\Theta}(\lambda_M) & G^c_{\mathcal{V}\mathcal{V}}(\lambda_M) & G^c_{\mathcal{V} P_g}(\lambda_M) & G^c_{\mathcal{V} Q_g}(\lambda_M) & \mathbf{0} & \mathbf{0} \\ G^c_{P_g\Theta}(\lambda_M) & G^c_{P_g\mathcal{V}}(\lambda_M) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ G^c_{Q_g\Theta}(\lambda_M) & G^c_{Q_g\mathcal{V}}(\lambda_M) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\right\} \\ + \Im\left\{\begin{bmatrix} G^c_{\Theta\Theta}(\lambda_N) & G^c_{\Theta\mathcal{V}}(\lambda_N) & G^c_{\Theta P_g}(\lambda_N) & G^c_{\Theta Q_g}(\lambda_N) & \mathbf{0} & \mathbf{0} \\ G^c_{\mathcal{V}\Theta}(\lambda_N) & G^c_{\mathcal{V}\mathcal{V}}(\lambda_N) & G^c_{\mathcal{V} P_g}(\lambda_N) & G^c_{\mathcal{V} Q_g}(\lambda_N) & \mathbf{0} & \mathbf{0} \\ G^c_{P_g\Theta}(\lambda_N) & G^c_{P_g\mathcal{V}}(\lambda_N) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ G^c_{Q_g\Theta}(\lambda_N) & G^c_{Q_g\mathcal{V}}(\lambda_N) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\right\} \end{aligned} \tag{153}$$

and

$$H_{XX}(\mu) = \begin{bmatrix} H^f_{\mathcal{X}\mathcal{X}}(\mu_f) + H^t_{\mathcal{X}\mathcal{X}}(\mu_t) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{154}$$

$$= \begin{bmatrix} H^f_{\Theta\Theta}(\mu_f) + H^t_{\Theta\Theta}(\mu_t) & H^f_{\Theta\mathcal{V}}(\mu_f) + H^t_{\Theta\mathcal{V}}(\mu_t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ H^f_{\mathcal{V}\Theta}(\mu_f) + H^t_{\mathcal{V}\Theta}(\mu_t) & H^f_{\mathcal{V}\mathcal{V}}(\mu_f) + H^t_{\mathcal{V}\mathcal{V}}(\mu_t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{155}$$

7.2 Nodal Power Balance

See the corresponding section in MATPOWER *Technical Note 2*.

8 Revision History

- **Revision 1** *(Jume 20, 2019)* – Added separate references for MATPOWER software and User's Manual. Included DOIs for all MATPOWER software, User's Manual and Technical Note references. Updated MATPOWER website links to `https://matpower.org`.
- **Initial version** *(April 2, 2018)* – Published as "MATPOWER *Technical Note 3*".

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